**On what epistemological thinking brings (or does not bring) to the analysis of tasks in terms of potentialities for mathematical learning.**

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As recalled by the proposed framework text, the relationship between the completion of tasks by students and the effectively achieved mathematical learning must be questioned. The present contribution will develop and illustrate that this relationship may be adequately analysed by adopting a praxeological perspective (Chevallard, 1999), supported by an epistemological study. Indeed, such a view allows to highlight the fundamental character of the task with respect to the targeted knowledge, and to identify the praxeological level adapted to the level of teaching (Schneider, 2009). Therefore, it allows a certain understanding of the behaviour of students, allowing to assess the impact of the tasks on the learning processes, beyond a possible methodology based on pre/post tests.

Keywords: epistemological obstacle, fundamental task, praxeology, praxeological level, pre/post test

Introduction

Teachers, researchers and the mathematical community in general have an interest in designing tasks to help students, pupils … acquire mathematical knowledge. This goal in mind, a first methodological problem arises regarding their efficiency. A common practice to evaluate the impact of a task is the use of pre/post tests, mimicking procedures found in (hard) sciences like biology and chemistry. The work of Brousseau (Brousseau, 1998) issues a strong warning towards such practices. Although they may be sound in other contexts, in the setting of mathematical education they are subject to great concerns because of sociological considerations pertaining to human behaviour in learning institutions. Brousseau have shown how a teacher and his pupils may engage in some sort of role playing where pupils decode, from the behavior of the teacher, or other hints not drawn from the knowledge they are supposed to acquire, how to answer the given task and how the teacher, consciously or not, gives credit to his pupils doing so, giving an overall illusion of knowledge acquisition. Those initials analysis have been shown to carry over different levels of education (Calmant, 2004, Job, 2011, Rouy, 2007).

Though it puts into perspective the pre/post tests methodology, the sociological viewpoint used by Brousseau might led some people to believe that mathematical understanding ultimately only relies on considerations where mathematical knowledge doesn’t need to be analysed e.g. an epistemological study of the targeted knowledge isn’t required. Far away from this conclusion, Brousseau, again, shows, thanks to the concept of epistemological obstacle, how much the inner peculiarities of mathematical knowledge are at the very heart of mathematical difficulties encountered by students and pupils, and, moreover, that the “toughness” of these difficulties leads, to some extent, to the aforementioned role playing between a teacher and his pupils, as a way of trying to do some mathematics despite the encountered difficulties. As a conclusion the work of Brousseau shows beyond any reasonable doubt that an epistemological analysis of the knowledge of concern cannot be bypassed in the design of mathematical task. But what kind of tool do we have to convey such an inquiry? How can we give credit to a task if the pre/post tests methodology is unsound? Those questions can be addressed, at least to some extent, using the praxeology concept.

Praxeologies, a model of (mathematical) knowledge

The work of Chevallard (Chevallard, 1999) extending some parts of Brousseau’s work endow us with the concept of praxeology, a model of what (mathematical) knowledge is, whose great strength is to address in a single coherent framework both anthropological and epistemological concerns mentioned above.

According to that theory, any activity, including mathematical ones, can be conceptualized as a task, something to do, a technique used to solve it, and a justification of the technique used to solve the task that can be split into a technology and a *theory*, a theory being a more abstract level of justification than the technology.

We will show in the sequel how praxeologies can be put to good use and among other things shed some more light onto the distinction between technology and theory. But before that we should emphasize that one breakthrough permitted by this approach is to allow mathematical knowledge to gain a form of relativism. This doesn’t mean that Pythagoras’ theorem will sometimes be true and other times false, but that different institutions may have different views on the same knowledge e.g. a given knowledge may be a simple technique in one institution, like the limit concept in many Belgian secondary schools, or a theory developed to give analysis a Euclidean architecture as has been done by Cauchy who in many ways can be considered the father of analysis and the creator of the modern concept of limit (Job, 2011).

Two different praxeological levels

This institutional relativity of mathematical knowledge has led Schneider (Schneider, 2009) to distinguish between two different kinds of praxeology, modelling ones and deductive ones, allowing us, as we shall explain, to understand the dynamic behind some hard to teach concepts like the limit one (Job, 2011), but also to design tasks that transcend the pre/post tests pitfalls.

In the first kind of praxeology, modelling ones, the fundamental task is to compute features of objects like areas of surfaces, volumes of solids whose existence doesn’t rely yet on a formal definition. Those objects exist as mental constructions shared, or believed to be, by some institutions. Justifications given in those praxeologies to techniques developed to address the fundamental task often rely on pragmatic arguments. A technique is validated if the results obtained are in accordance with results derived using other valid techniques that may even belong to other fields of sciences. For instance, early infinitesimals techniques where used and recognized based on the accordance with results obtained using physical arguments of cinematic nature.

In the second kind of praxeology, deductive ones, the fundamental task consist in defining those mental constructions, to make explicit what was left in the shadow in the first kind of praxeology and built a deductive theory. Often the techniques used in modelling praxeology are used in deductive ones as definitions. The definition of integral given by Cauchy is a good example of such a procedure: an approximation procedure is turned into a definition that, in turn, is used to prove theorems about integrals (existence, uniqueness ...).

The two kinds of praxeology are distinct but closely related, the second kind often taking place after the first one. With these two concepts we don’t aim, nor claim to encompass the whole mathematics, but important parts of its growth like the birth of analysis from calculus. Indeed, it can be showed that calculus can be roughly speaking represented as a modelling praxeology and analysis a deductive one (Job, 2011). These two praxeologies take into account the institutional relativity of various concepts like the limit one and the derivative that may otherwise be seen as concepts that somehow where born at some place in time almost as they appear nowadays whereas they evolved under the guidance of very different viewpoints. The calculus period was mostly guided by the will to be able to compute areas… and analysis was created out of the will to purge calculus from geometry and physics, forge a new area of mathematics whose rigor would equal that of the ancient Greeks.

Epistemological thickness and fundamental character of a task

The praxeological levels introduced above allow us to get back to our initial asking. How do we assess a task? A partial answer given by Brousseau (Brousseau, 1998) is to consider a task *fundamental* with respect to a given knowledge if the knowledge takes places in a praxeology as a technique where the task cannot be solved without that knowledge. The knowledge is thus seen as a kind of optimal answer to the proposed task. This requirement is legitimated by what we have said earlier in this article about the role playing pupils, students and teachers are prone to engage. It shouldn’t be possible to solve the task used to teach a certain knowledge only using hints external to the knowledge like the teachers eyebrows indicating if the students are running along the required lines.

This understanding of the fundamental nature of a task has been shown to be effective to introduce concepts likes the rational numbers (Brousseau, 1998) but doesn’t seem to translate well to concepts like the limit one. Indeed, it is one of the twentieth century achievements to have shown with the work of Abraham Robinson that a sound basis could be given to infinitesimal concepts so far rejected as a sound basis for calculus. The limit concept is thus by no means necessary to cast the calculus into a deductive mould.

Anyway, the very heart of Brousseau’s idea can be adapted is the following manner, taking into account the institutional relativity of knowledge introduced above. A task is said to be fundamental (in a broad sense) with respect to a given knowledge and a given institution if that institution takes for granted the knowledge is optimal to solve the task. In this new definition there is no more necessity in a “mathematical” way but an anthropological necessity that an institution gives to itself.

At this point, we are now able to understand the leading role played by praxeological levels. The structure of a fundamental task and even the fundamental character of that task with respect to, for instance, the limit concept depend on what kind of praxeology we place ourselves in. A fundamental task for the limit concept in a deductive praxeology won’t be the same as a fundamental task in a modelling one. Before we dive into some characteristics of these tasks, let us first give an example of the consequences of not being able to clearly state whether a task belong to one praxeological level or the other.

The consequences of blurred praxeological levels in secondary school

In (Job, 2011) we study the teaching of the limit concept in secondary school and are able to support the following views. Secondary school tries to teach the limit concept but fails to do so, unable to identify the praxeological level where it should belong.

Secondary school tries to teach this concept giving students elements that belongs to the deductive praxeology of analysis mathematicians use nowadays in order to place itself under the supervision of that institution from which it draws its legitimacy. This deductive praxeology being out of reach to students of that age, the school praxeology mainly consists of elements acting as blazons, that is, parts of the original praxeology that are able to support the illusion of a real teaching of the limit concept from an outside perspective.

Among these blazons, the definition of a limit plays a key role. Secondary school tries to teach this definition using various tricks to make believe students this definition is a somewhat complicated (mathematical) way of saying something very natural. For instance, it gives students tables with values of x and f(x) for a given function, waiting for the students to recognized some sort of behaviour that should be put into sentences like “as x tends to … f(x) approaches …”. Starting from such sentences, teachers gradually turn these into the required forms “f(x) can be made as close as one wishes to …” using arguments that belong more to rhetoric than mathematics.

Such an approach is misleading in nature for the definition of the limit concept was designed by Cauchy *to conduct proofs* and define other key concepts of analysis like the derivative. But except for a few trivial ones, proofs in secondary school are left aside. So the very use of the limit concept in the deductive praxeology where it belongs is left aside. The school praxeology thus bears no fundamental character whatsoever.

Such a fool’s game isn’t the consequence of any malicious thoughts on the side of secondary school but the resultant of antagonist constraints. On the one hand, it has to teach the limit concept in a way mathematicians would recognize as valid, which is a daunting task. On the other hand it must succeed in that task. The only way secondary school has to its disposal to conciliate the two is to take the deductive praxeology, strip it from most of its content and wrap it in a discourse that can be accepted by students even if the cost is to propose tasks that have no fundamental character. This wrapping is partly a consequence of its unawareness of the existence of another praxeology (a modelling one) where the limit concept is legitimate.

So secondary school’s praxeology with respect to the limit concept lies in a no man’s land, not being in a deductive or in a modelling praxeology. Similar conclusions are drawn in (Rouy, 2007) regarding the derivative also based on praxeological considerations. This analysis sheds a new light on the pre/post-tests methodology. How could we give credit to a task succeeding a sequence of pre and post tests if that task isn’t epistemologically consistent?

What praxeological level for secondary school?

The section above asks a crucial question. Is there a place left for the limit concept in school that would be mathematically legitimate? The answer might be positive if we place ourselves in a modelling praxeology. Although Schneider is critical towards some of the tasks they designed (Schneider, 2001), AHA (AHA, 1999) has proposed a fundamental task for the limit concept in a modelling praxeology, which is declined at the various levels of application of the concept in sub-tasks (areas, speeds, tangents).

On the other hand, Job has studied the teaching of the limit concept in a deductive praxeology (Job, 2011). Its results show how much a deductive approach to the limit concept is a very demanding task. In a few words, the students were asked to propose definitions of a certain behaviour of sequences of real numbers and then to proof properties related to this behaviour. The students were mostly unable to make their definitions evolve. They stayed stuck with definitions that are “descriptions” of what they see of the studied behaviour. They couldn’t possibly envision their definitions as something to be chosen to allow proofs despite the many contradictions pointed out by the teacher. This inability is related to epistemological obstacles. Students see definition as a description of some mental concept they believe everyone of them share. They therefore don’t understand the rules of the game they are asked to play, feeling they are asked something unnecessary complicated because “everyone agree with the found properties”, “nothing has to be proved”. This situation seems like a dead end because the teaching school has given them tends to reinforce their vision of mathematics, depriving them from the need to cast theories into a deductive mould.

Different understandings of the task concept

Let us give a second example of the use of praxeologies that will put the task concept itself into question, showing it should sometimes be understood at a different level than is usually done, thus clarifying the concept of a fundamental task understood in a broad sense.

We shall illustrate our views through a task used by our team (Job & Schneider, to be published) to teach negative numbers and specifically the multiplication rule to 12 years old pupils. Being as concise as possible, pupils are asked to devise a single formula that allows them to encompass the motion of two vehicles, being flashed by a radar, driving different roads, but at the same constant speed of 2km/min. A first formula *p=2t* emerges for positive times where *p* denotes a location and *t* a time[[1]](#footnote-1). They are then asked to elaborate a formula that would also be valid for negative times e.g. times before the two cars are flashed. This requirement of a single formula brings pupils face to face with expressions like -6=2 x (-3) and therefore to an extension of the multiplication rule for positive numbers to negative ones. Pupils are then asked to deal with cars driving in the direction opposite to the one the first two where driving. This introduces “negative” speeds, the minus sign telling which direction the car is driving. The same requirement of a single formula leads in turn to expressions like 6=(-2) x (-3) a completes the multiplication rule for negative numbers.

Such an introduction of negative numbers and their multiplication meets pupils’ global assent but what we are trying to emphasize lies somewhere else. The peculiarity of our task doesn’t rely so much on the pupils’ assent, but on a characteristic where they are not involved in the first place. This task tries to expose pupils to a choice made by mathematicians/physicists to allow them to model with a single formula the various incarnations of the same motion, in terms pupils should be able to understand. Pushing the structure of our task to the extreme, it doesn’t matter so much if the pupils agree with the decision made by mathematicians/physicists as long as they understand there is a choice to be made and its consequences, because it is not their assent we are seeking. We simply try to make as explicit as possible choices made by some institution they have no impact on. Learning mathematics and physics also means learning the conventions of those institutions whether we agree or not with them. It is not to say that pupils have nothing to understand. On the opposite, there is something to understand which is located at a level that is subtle to explain, not to pupils who are living the task, but to the mathematical learning community: if you want to learn mathematics you have to accept its conventions whether or not you agree with them as long as you understand why those conventions have been adopted.

Conclusion

We have argued that a pre/post-tests methodology is unsound to assess the efficiency of a task and that the distinction between two kinds of praxeologies (modelling and deductive ones) plays a key role in designing tasks and understanding the dynamic of ordinary lessons. A task should clearly identify whether it belongs to one praxeology or the other in order to be meaningful. A task that doesn’t belong to any of those two levels should be handled with great care, its fundamental character being dubious. Being able to state to what kind of praxeology we belong allow us to interpret students’ work in the light of a solid epistemological background, therefore giving us tools to avoid misinterpretations that pre/post-tests a prone to commit due to their very structure: a post test result better than a pre-test one isn’t obviously a sign of better understanding but may only be the result of an accommodation from the students that have understood how to answer the tasks without using the targeted mathematical knowledge. Taking advantage of the distinction made between modelling and deductive praxeologies and the relativity of knowledge, we have put into question the very concept of a task showing how much its understanding can be and should broadened as soon as we are dealing with the teaching of concepts like the limit one or the negative numbers. Those tasks should be understood in a broader sense than usual, as a way to highlight choices made by an institution and the reasons underlying these choices.

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1. Aside the multiplication rule, the task allow us to make pupils distinguish between distance and position among many other things we have no space to elaborate on. [↑](#footnote-ref-1)